Viability and resilience

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PATRES
PATTERN RESILIENCE
EU FRAMEWORK 6 NEST PROJECT

Outline

- **Resilience based on attractors**
- **Resilience based on viability**
  - Without management actions and attractors in desired set of states
  - With actions and attractors in desired set of states
  - With actions and attractors not in desired set of states
- **Conclusion**

Resilience of based on attractors

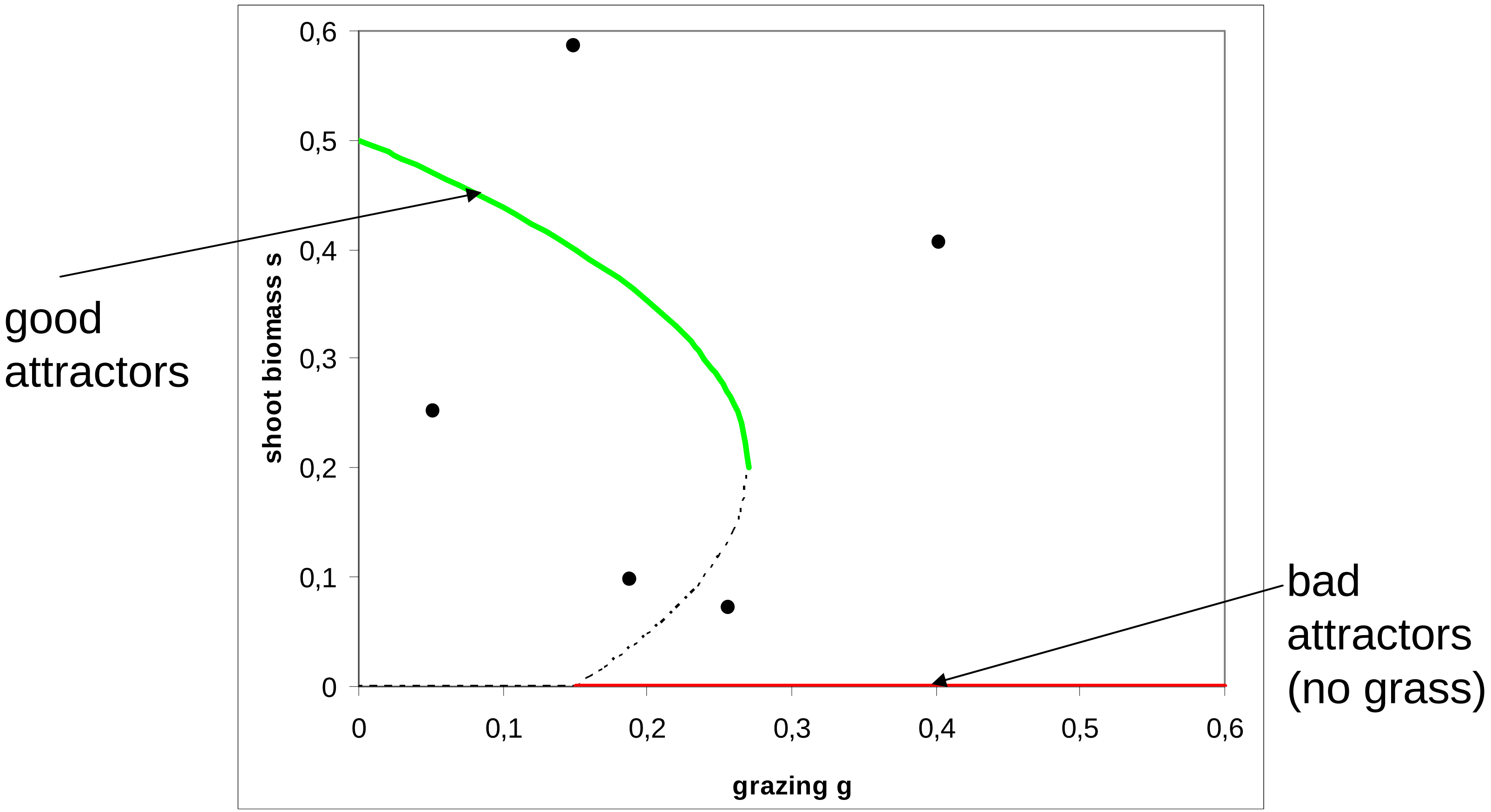
- **Hypothesis:**
  - some attractors provide desired properties of the system (good attractors)
  - some attractors don't (bad attractors).
- **The system is resilient to a perturbation if the perturbation keeps the system in the attraction basin of a « good » attractor, it is not resilient if the perturbation drives the system to the attractor basin of a « bad » attractor.**
- **Resilience value connected with the size of the « good » attractor basins.**

Simplified example of savanna dynamics

- **Taken from Anderies et al. 2003**
- **Two variables:**
  - shoot biomass (grass) *s*
  - grazing *g*
- **We suppose that once the value of grazing *g* is decided, it remains fixed.**

$$\frac{ds}{dt} = (\alpha s + \beta s^2)(1 - s - \gamma) - gs$$

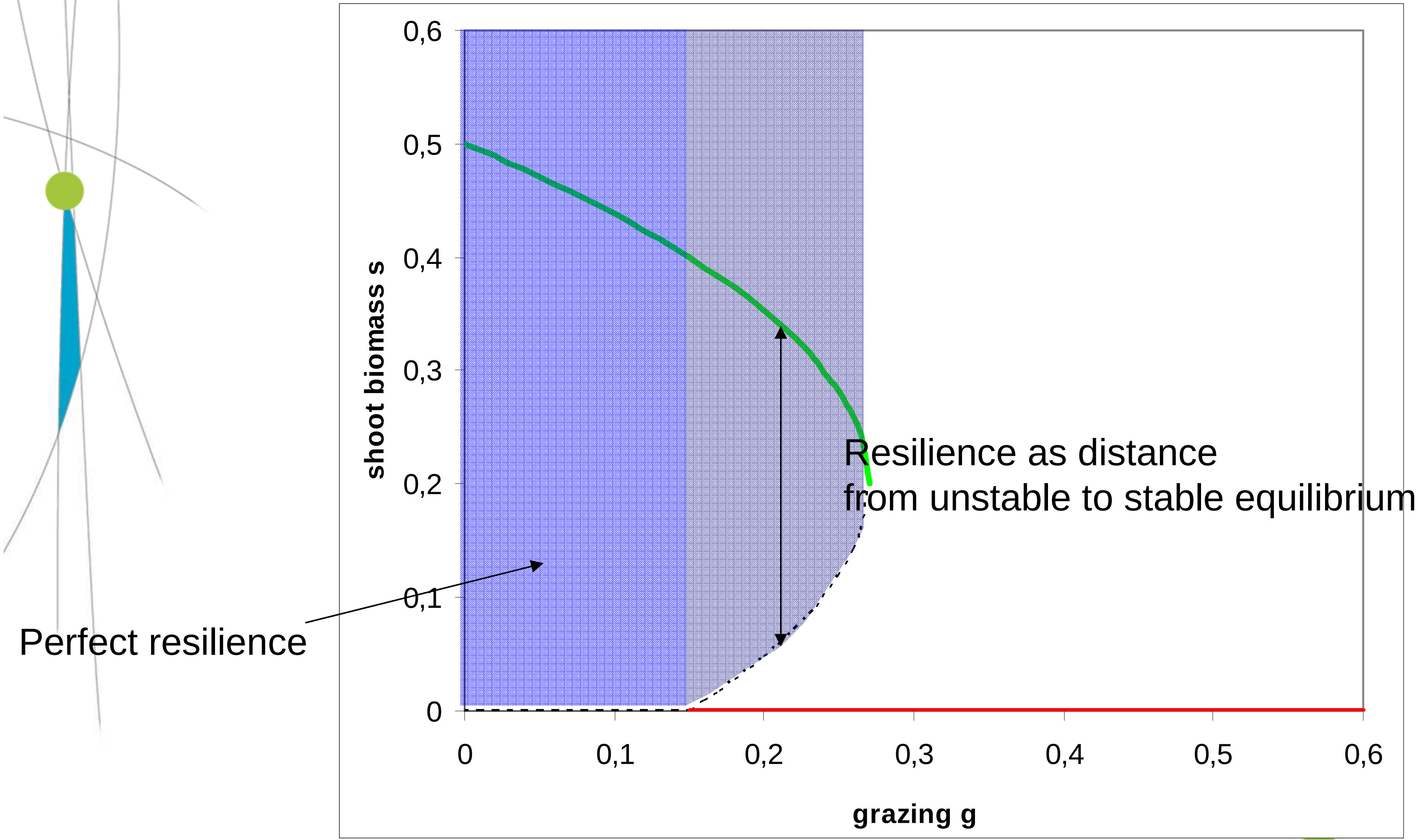
Good and bad attractors



Resilience based on attractors

- Dynamical system defined for instance in discrete time:
- $\mathbf{x}(t+dt) = \mathbf{x}(t) + \Phi(\mathbf{x}(t))dt$
- A point $\mathbf{x}_a$ is an attractor if:
  - $\Phi(\mathbf{x}_a) = 0$
  - There exists a subset A such that, for all $\mathbf{x}(0)$ in A:
  - $\mathbf{x}(t) \rightarrow \mathbf{x}_a$ when $t \rightarrow \infty$
  - The largest set *A* is the attraction basin
- **Resilience defined as**
  - the size of the attractor basin
  - the velocity for going to the attractor (based on a linearisation of the dynamics close to the equilibrium)

Definition of resilience as a function of grazing



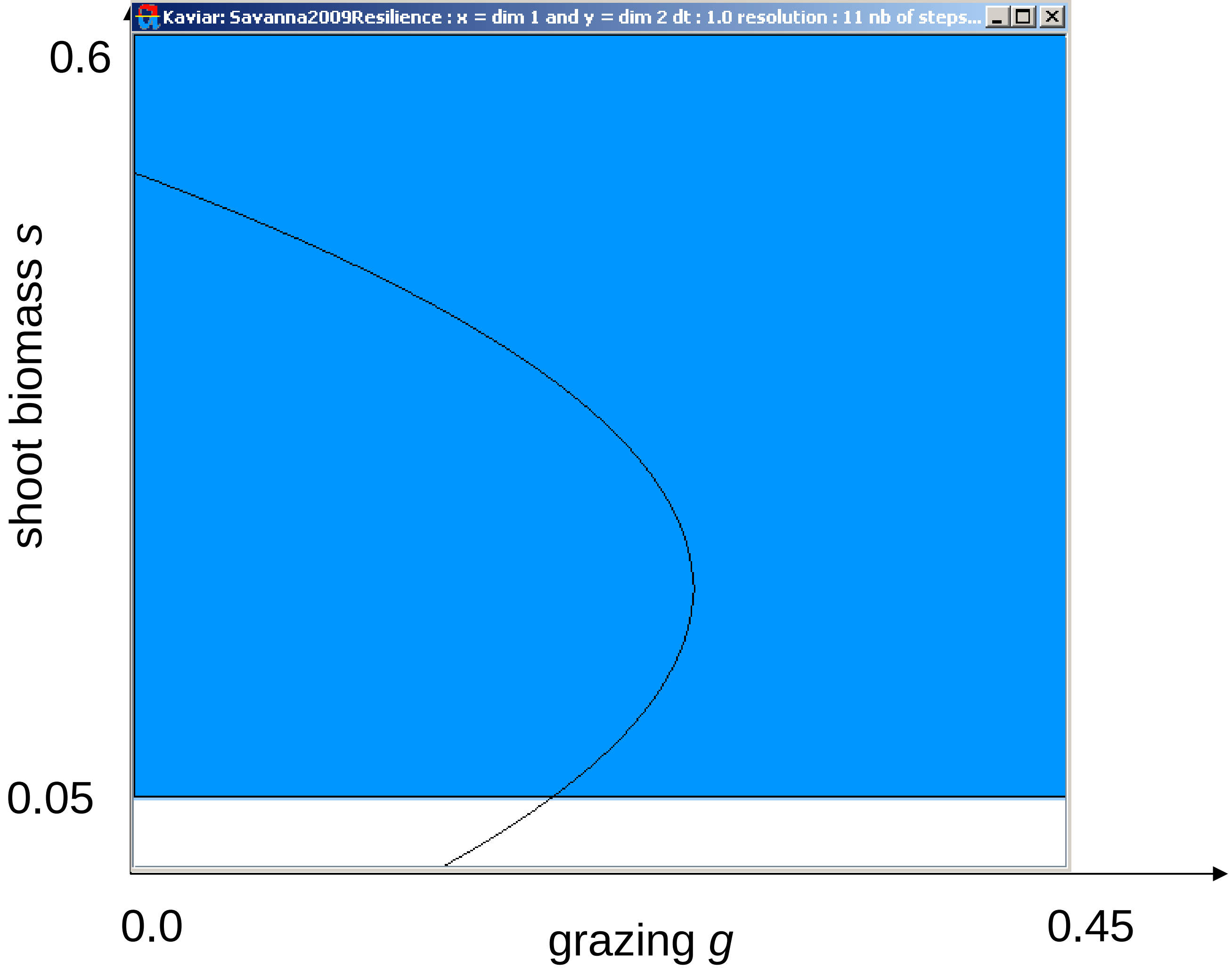
Limits of the attractor based resilience

- **Need to define the desired functioning of the system as a set of attractors.**
- **Difficult to introduce a management policy (it is supposed included in the system dynamics)**

Different view: Desired property as state set

- **Resilience supposes to define a desired property (functioning) of the system**
- **Main idea: define the desired property as as subset of the state space (independently from the presence of attractors)**

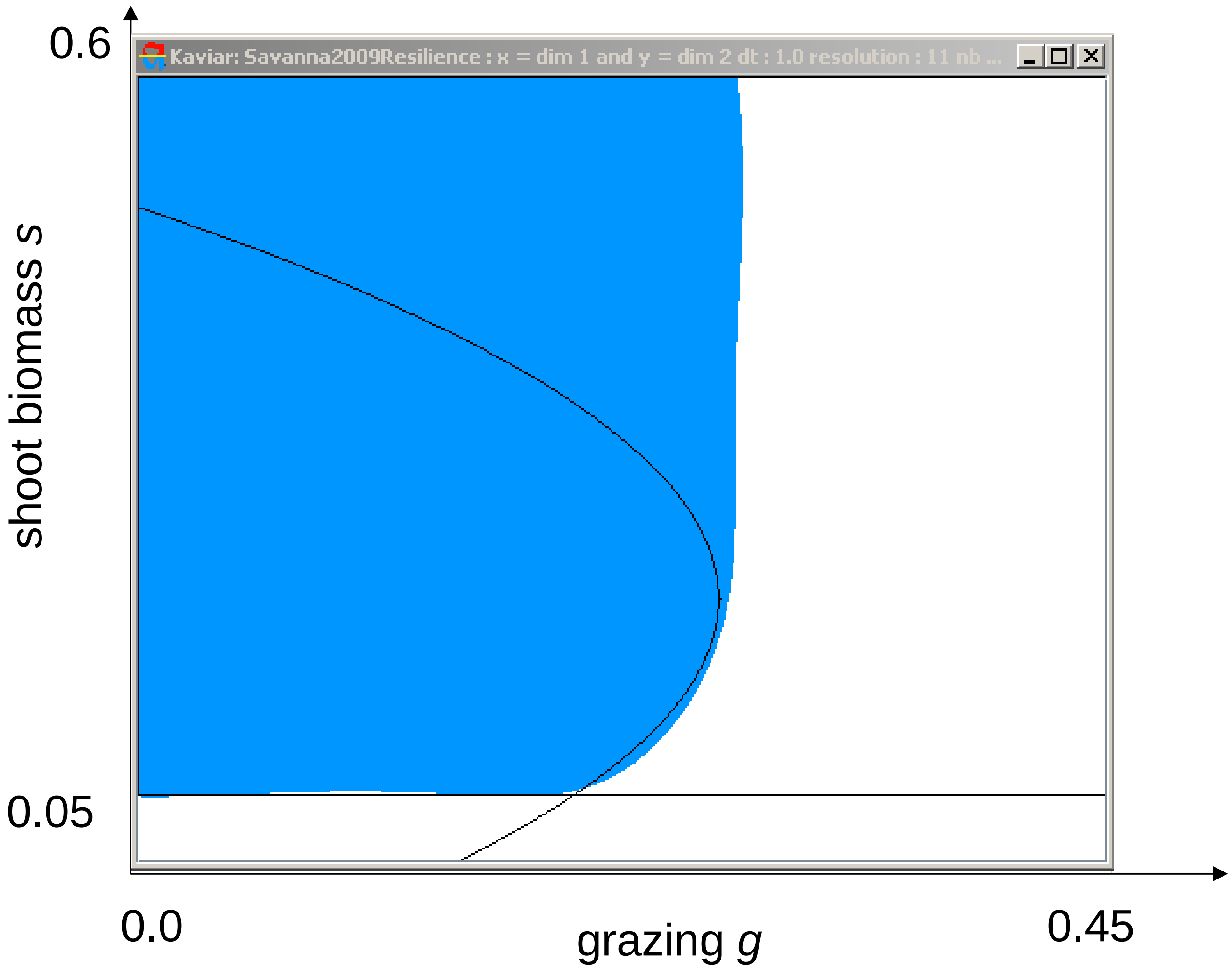
Constraint set: *s* > 0;05



Link with viability

- **We need to determine all points of space from which the desired property is maintained – i.e. which trajectory stays in the desired set**
- **Viability theory, developed in the 90ies by J.P. Aubin:**
  - considers a system that collapses or badly deteriorates if it goes beyond a state subset ***K***.
  - It needs also to determine the points from which the trajectory remains in ***K***, that is $x(\mathbf{0})$ such that, for all value t, we have $x(t)$ in ***K***, is called the viability kernel.

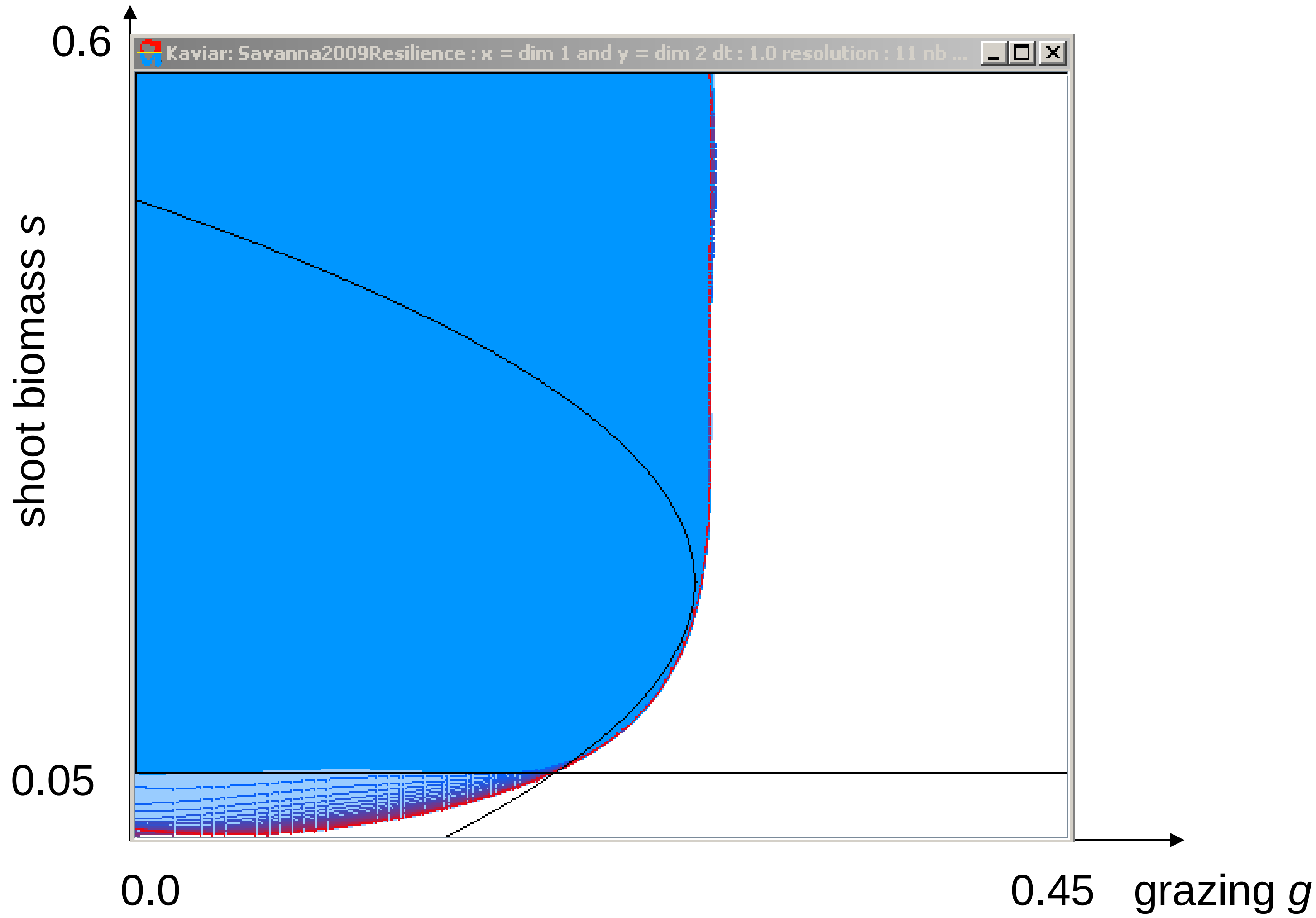
Viability kernel



Resilience based on viability (without action)

- **We also need to determine the points from which trajectories come back to the desired set and stay there.**
- **A resilient state is a state from which the trajectory goes to the viability kernel (because it guarantees it will stay there)**
- **In viability theory, the set of points from which trajectories go to a target set is the capture basin of this target set.**
- **The set of resilient states is the capture basin of the viability kernel**
- **The measure of resilience is the inverse of the integral of a cost per unit of time along the trajectrory to the viability kernel.**

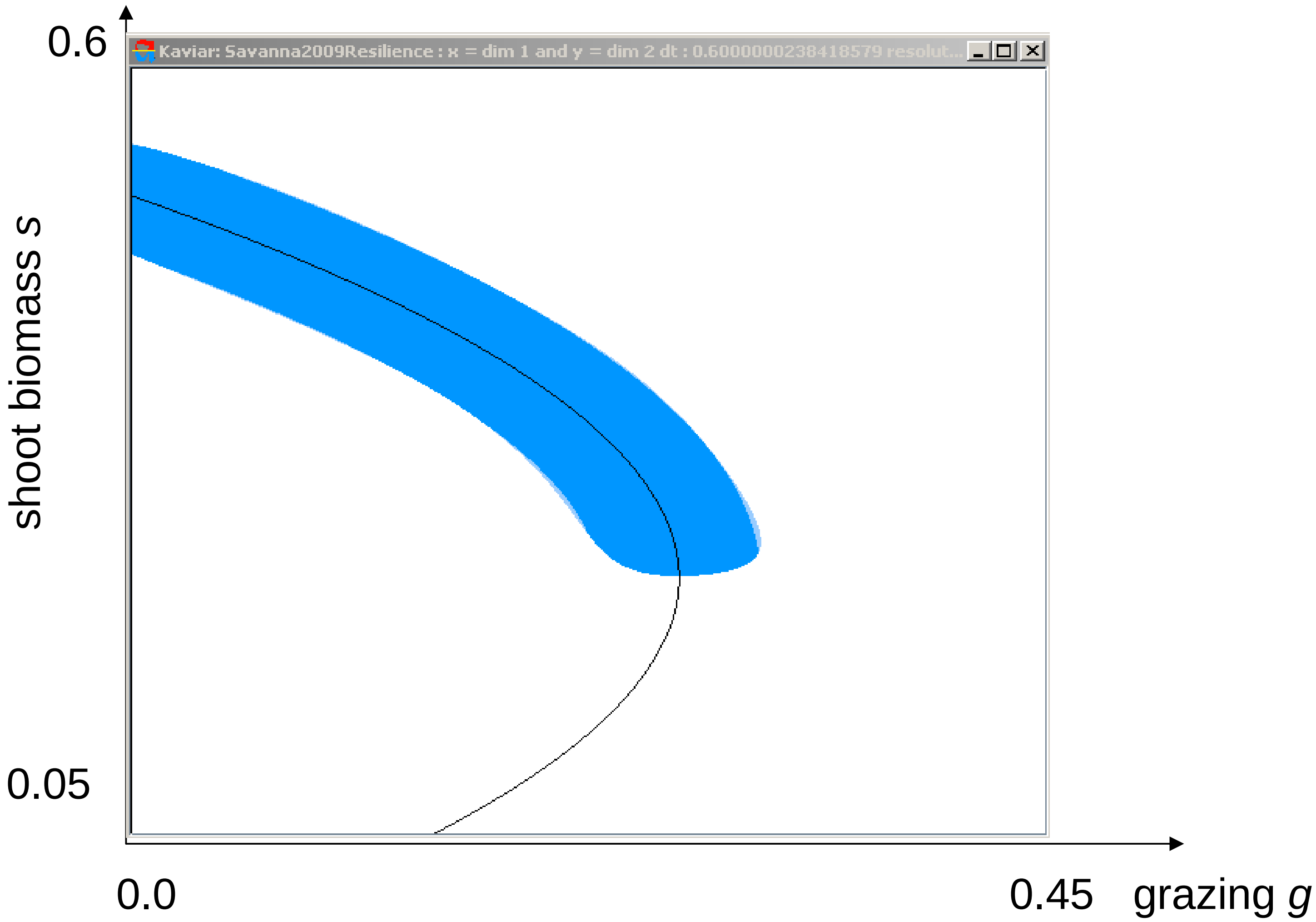
Resilience


0.6
Kaviar: Savanna2009Resilience : x = dim 1 and y = dim 2 dt : 1.0 resolution : 11 nb ...
shoot biomass *s*
0.05
0.0
0.45
grazing *g*

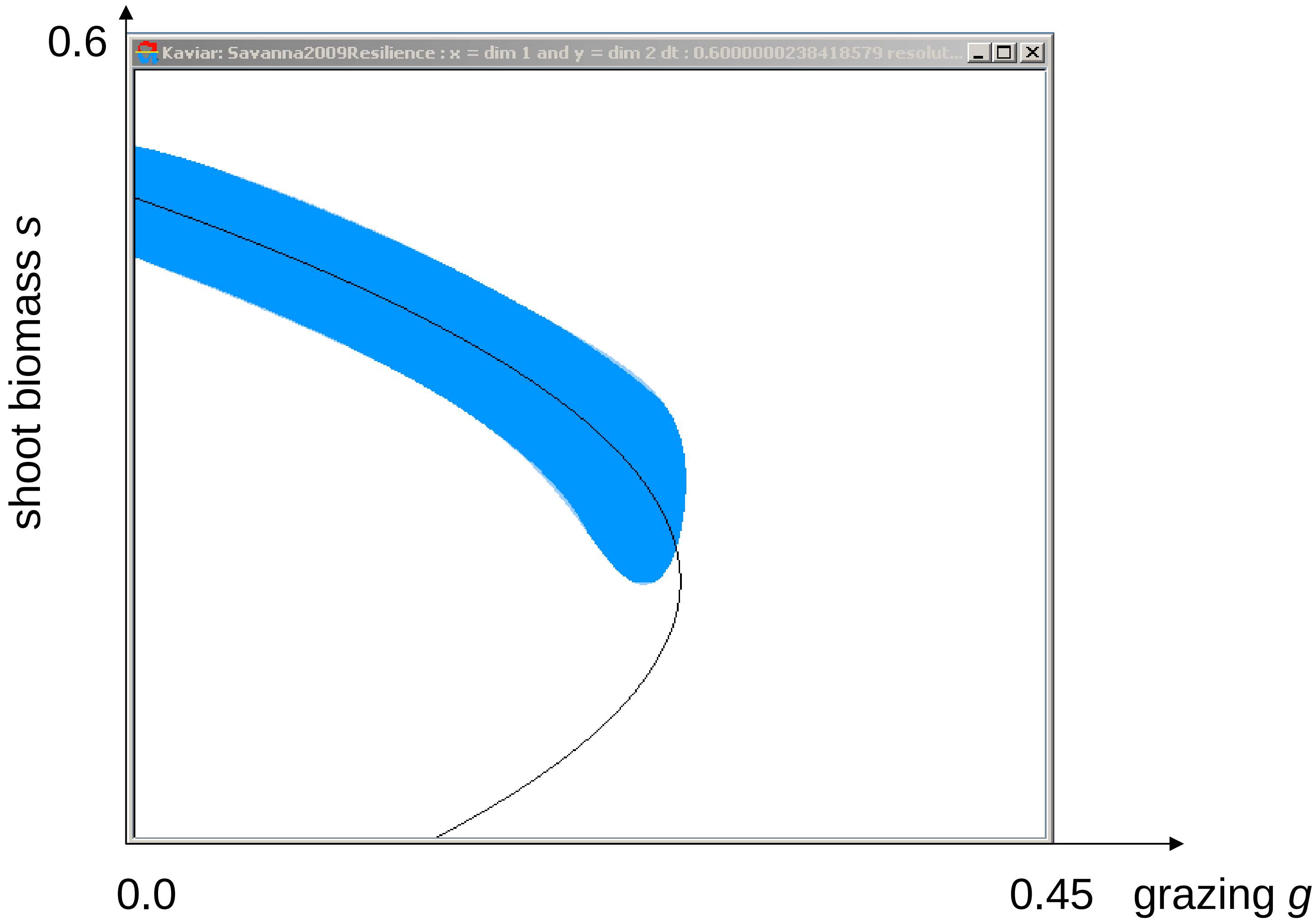
Comparing the definitions

- **Most states which are *resilient* in the attractor definition are *viable* in the viability definition**
- **The measure of resilience is different (depends on the dynamics)**
- **Difference due to the choice of the constraint set ?**

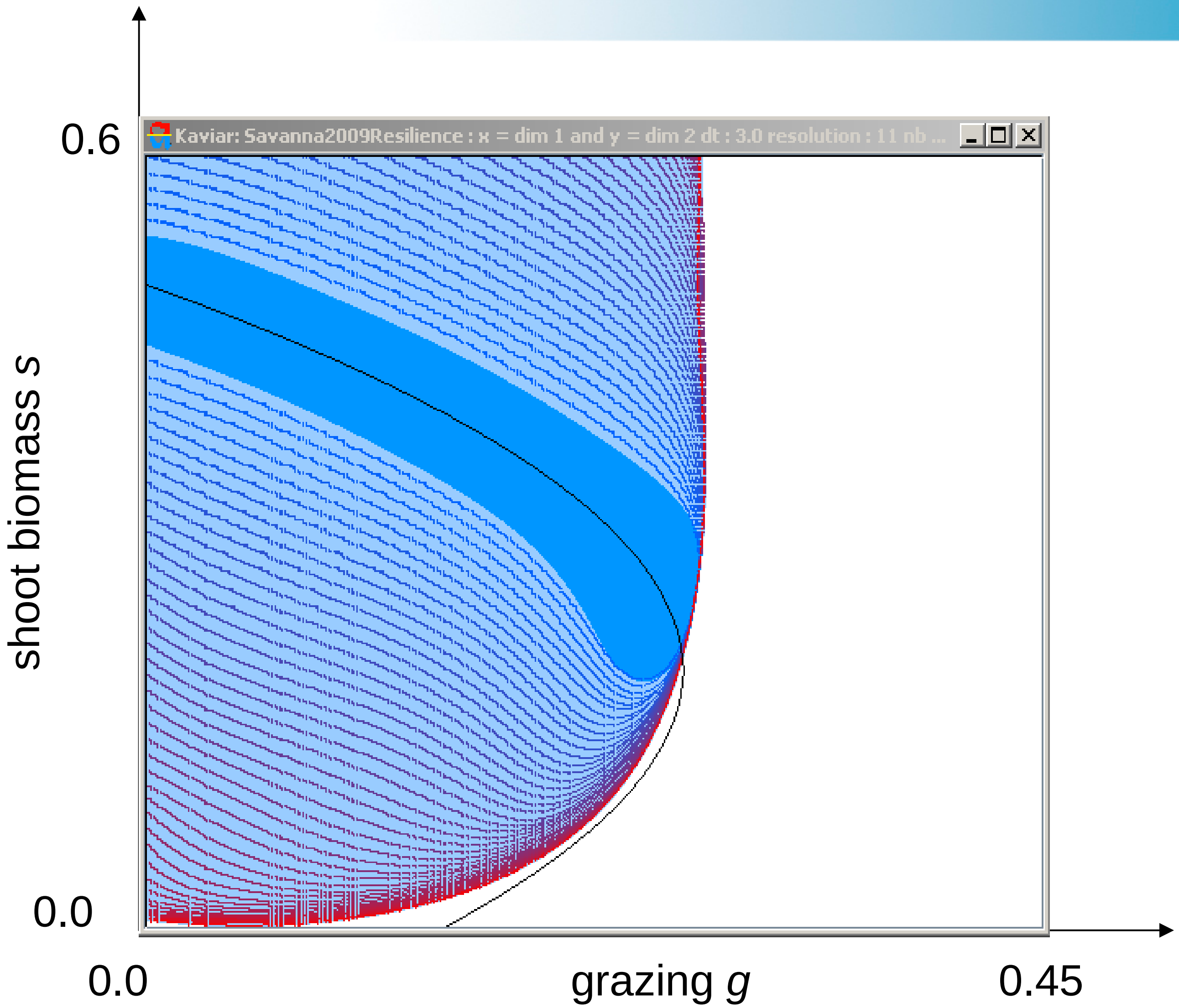
New constraint set



Viability kernel 2



Resilience 2



Comparing the definitions

- **The resilient states almost coincide.**
- **The resilience values are close but in some places depend more on the dynamics in the resilience viability definition.**

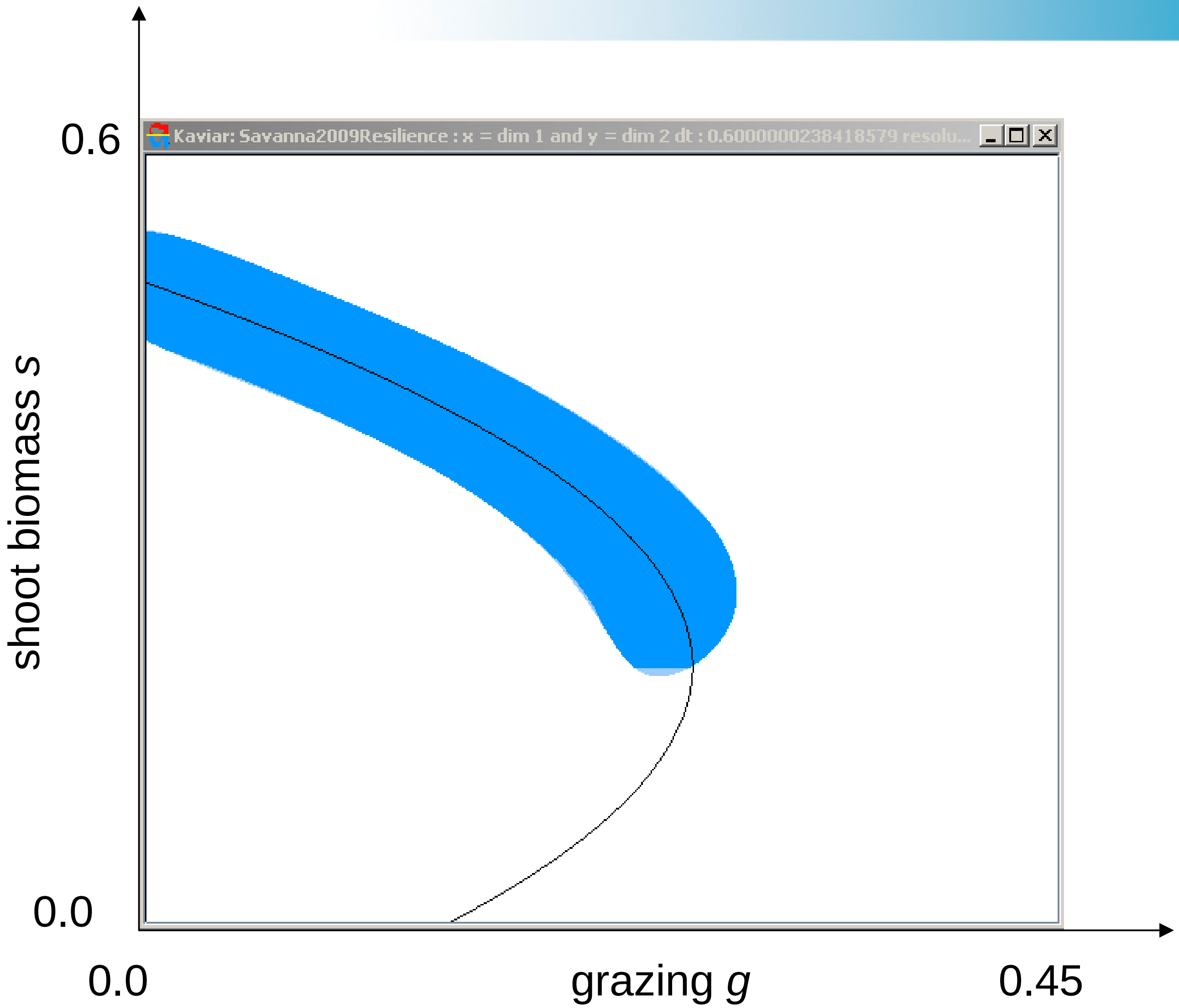
Introducing a possibility to act on the system

- **At each time step, we suppose that it is possible to act on the system.**
- **For instance, we suppose that the grazing pressure can be modified of a value dg, with -0.02 < dg < 0.02**

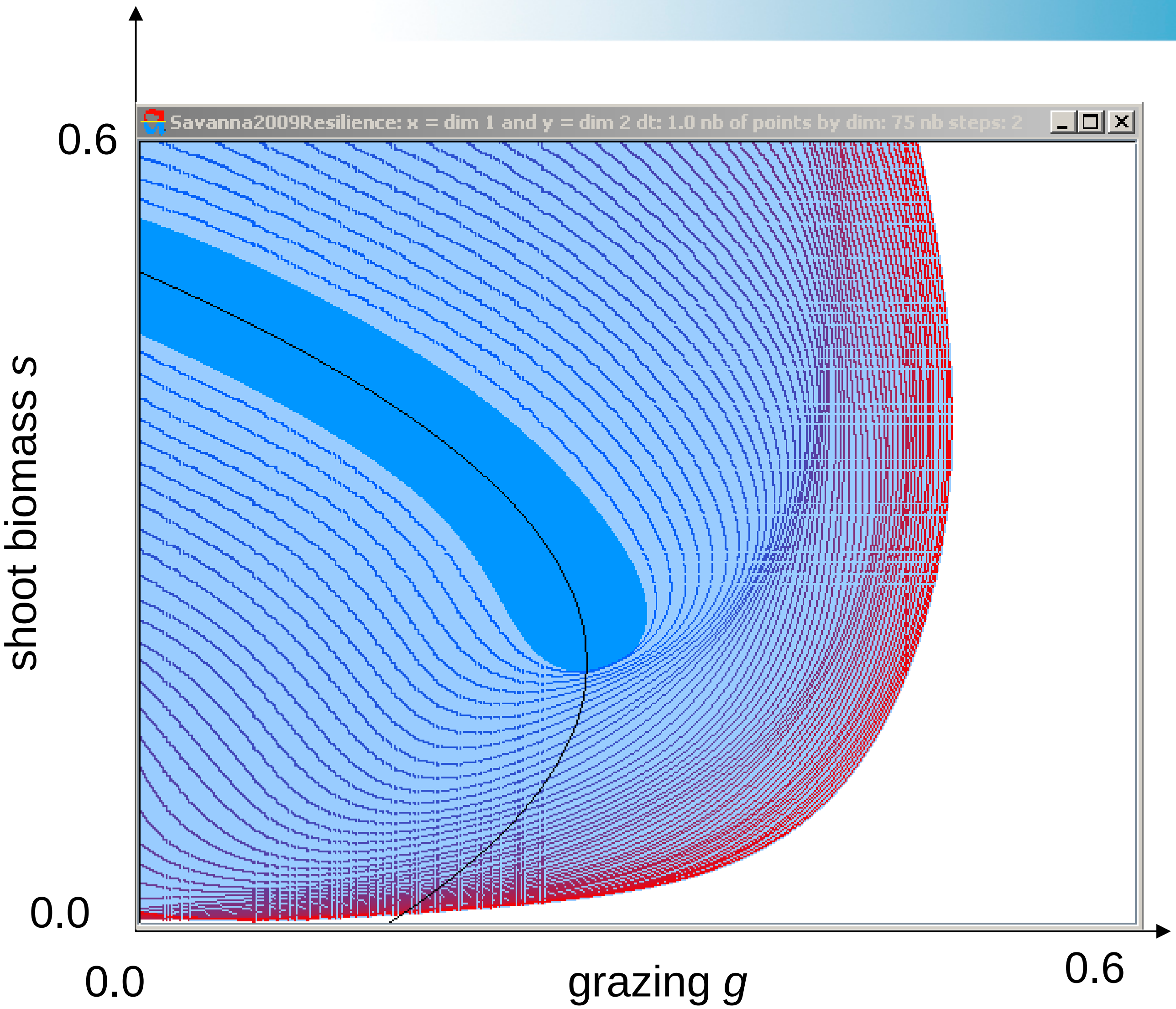
Viability theory with action

- **Viability kernel : the set of states from which a policy of action keeps the system within the constraint set.**
- **Capture basin of a target set: set of states from which there exists a policy of action leading to the target.**
- **Resilient states: states belonging to the capture basin of the viability kernel.**
- **Resilience value: inverse of the time to go back to the viability kernel.**
- **There exist general algorithms to compute viability kernels and capture basins**

Viability kernel 3

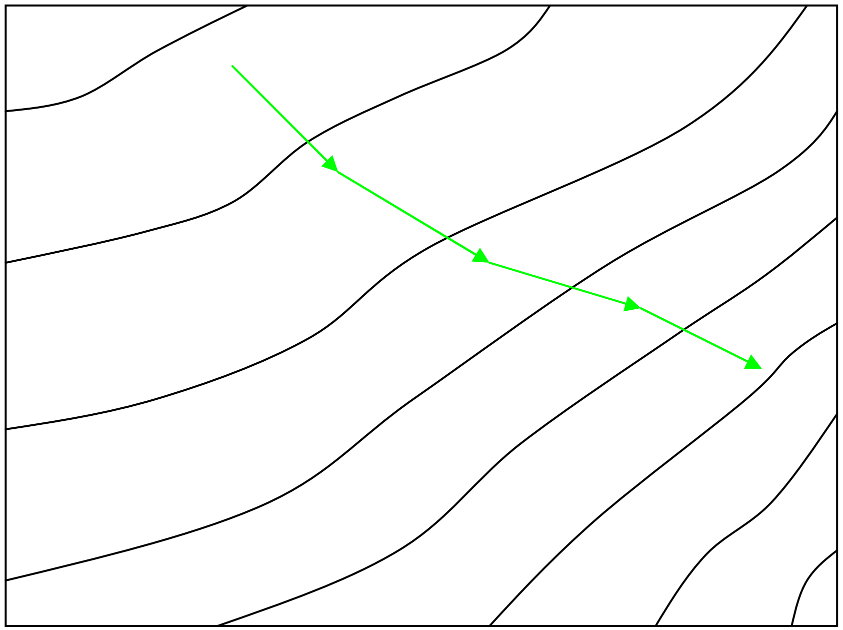


Resilience 3

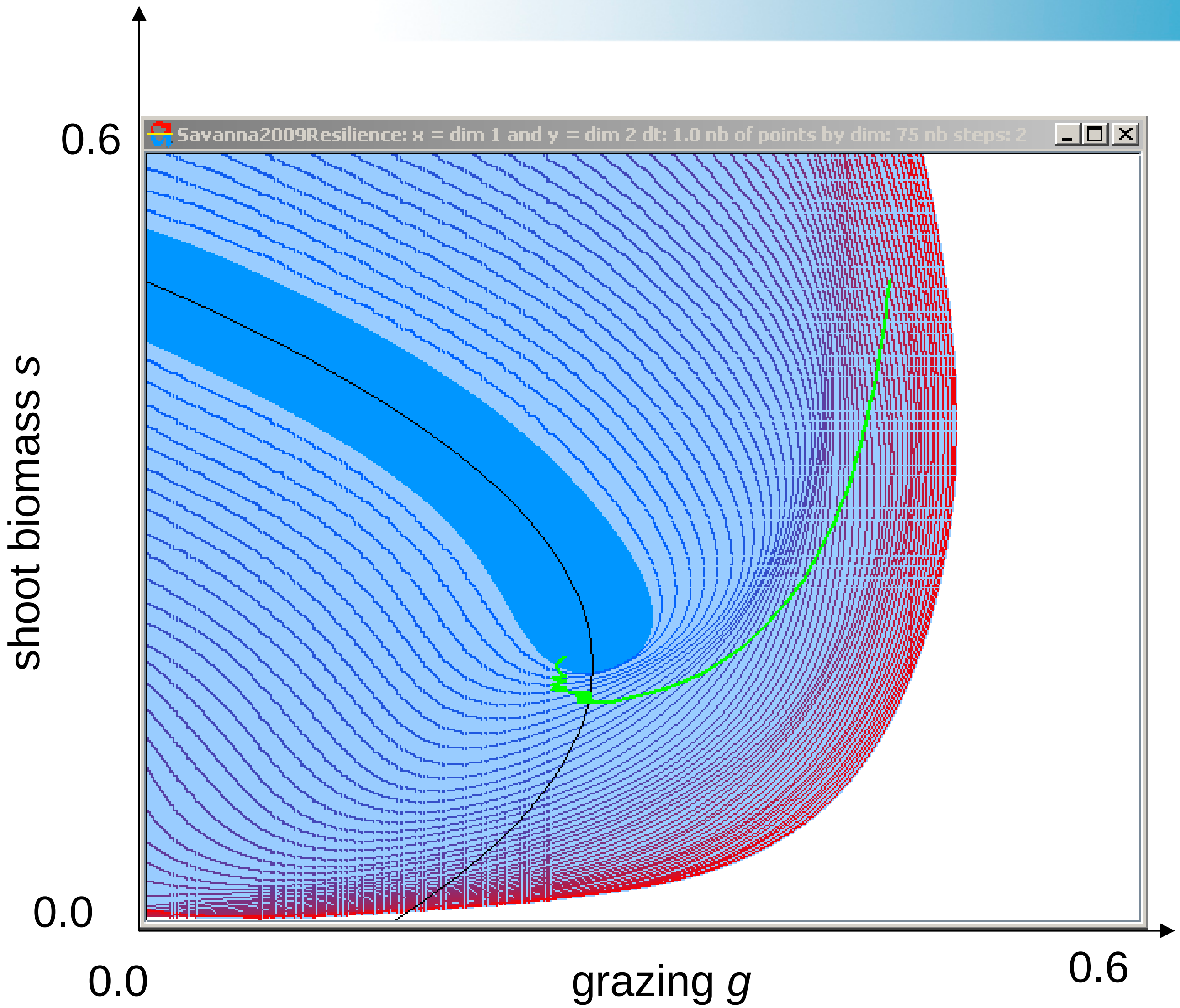


Defining actions

The action is the one that drives the system as close as possible in the normal direction of the next level line.



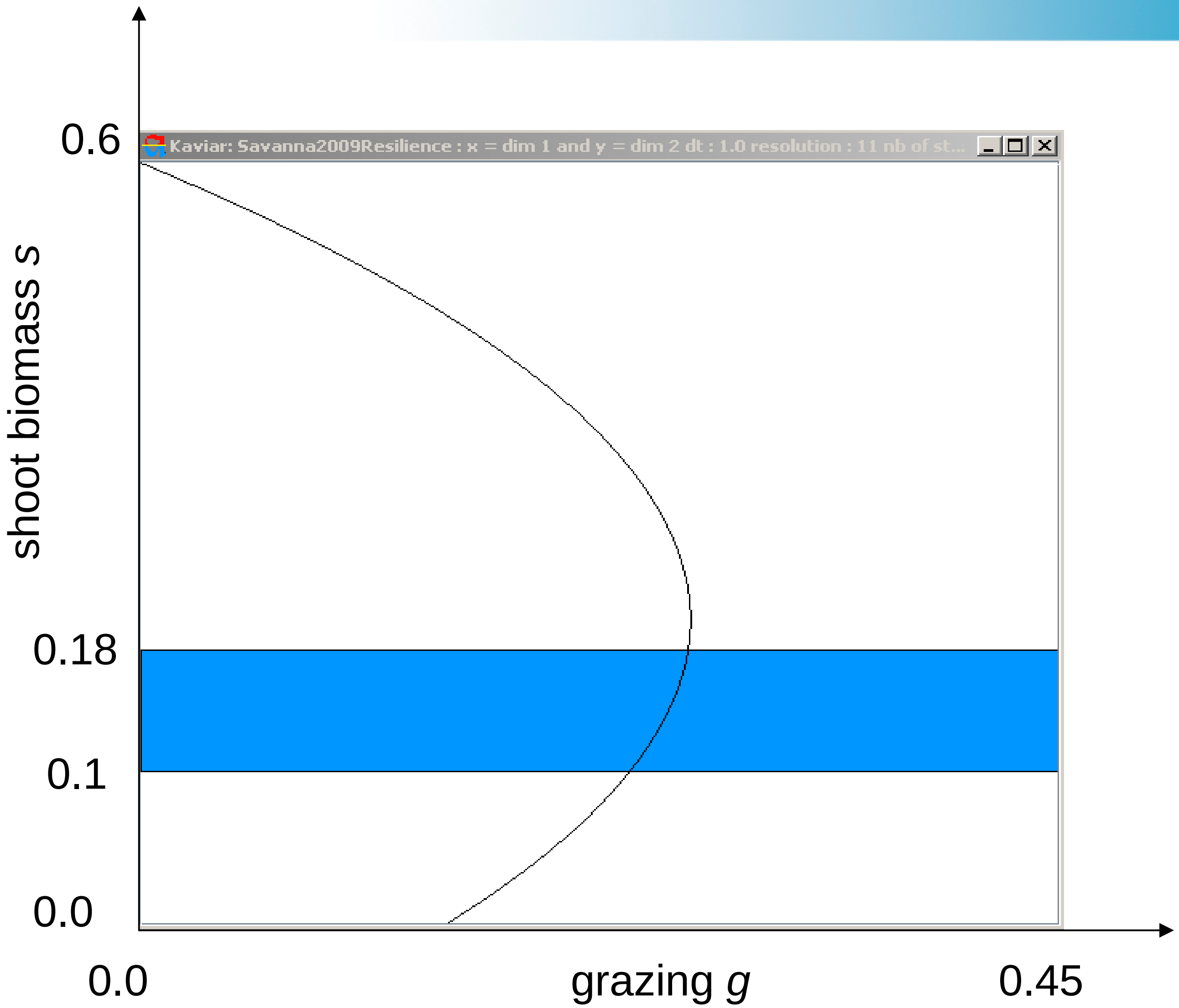
Resilient trajectory



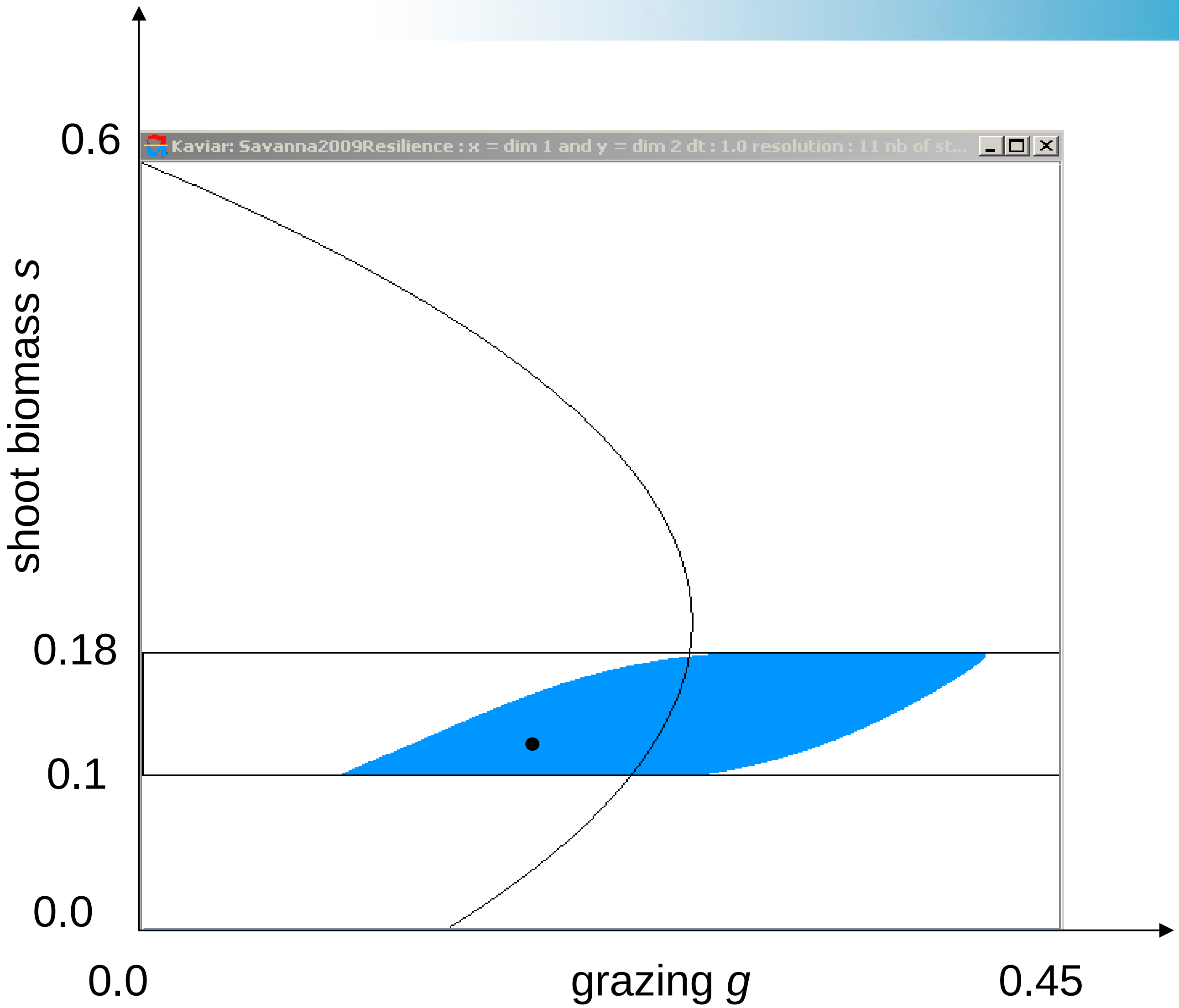
No attractor in the constraint set

- **The dynamics do not necessary lead to an attractor.**
- **Suppose now that we want to keep the level of grass between 0.05 and 0.18**
- **We still can change the grazing of at most 0.02 (positive or negative)**

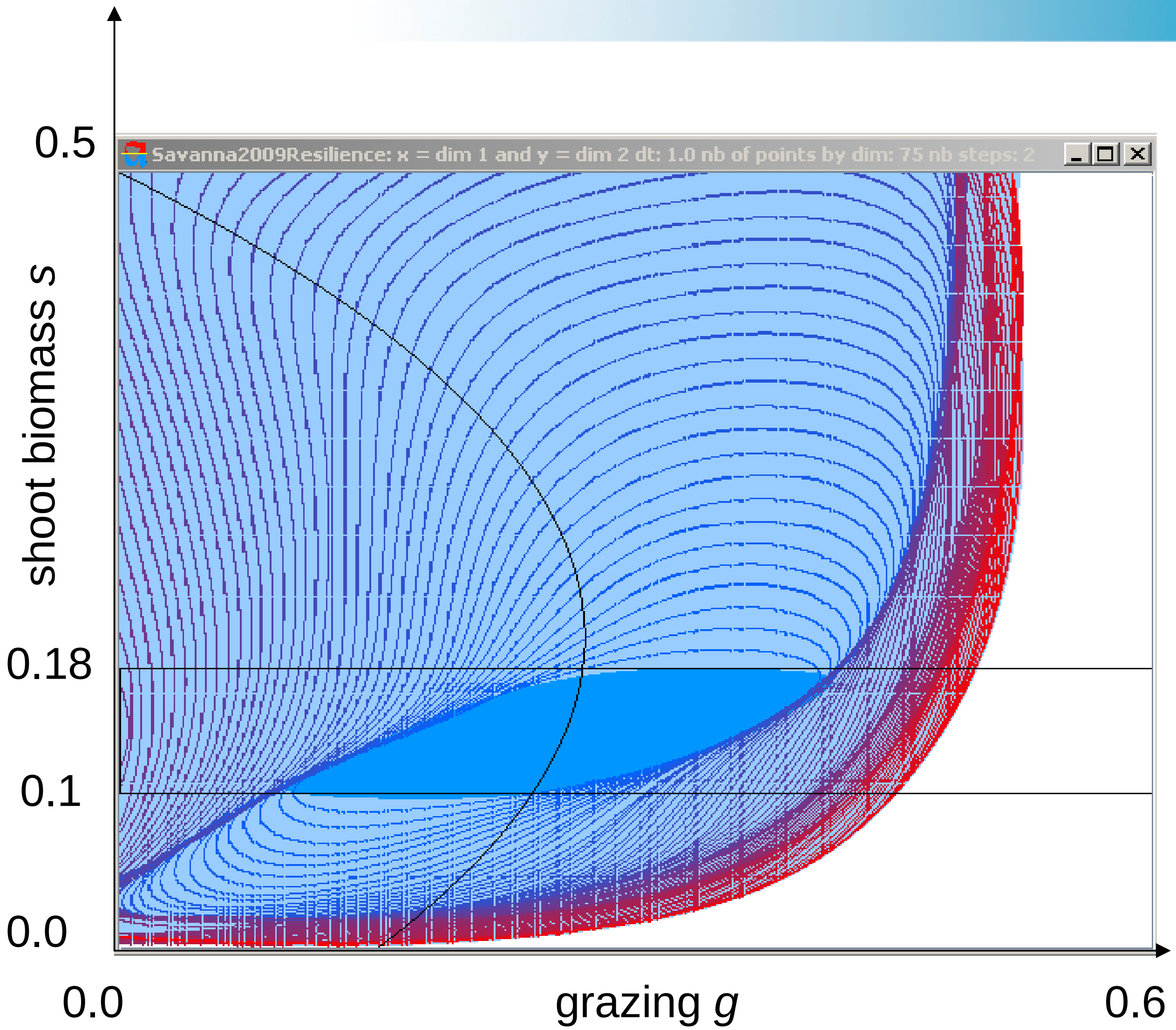
Constraint set : 0.1 < *s* < 0.18



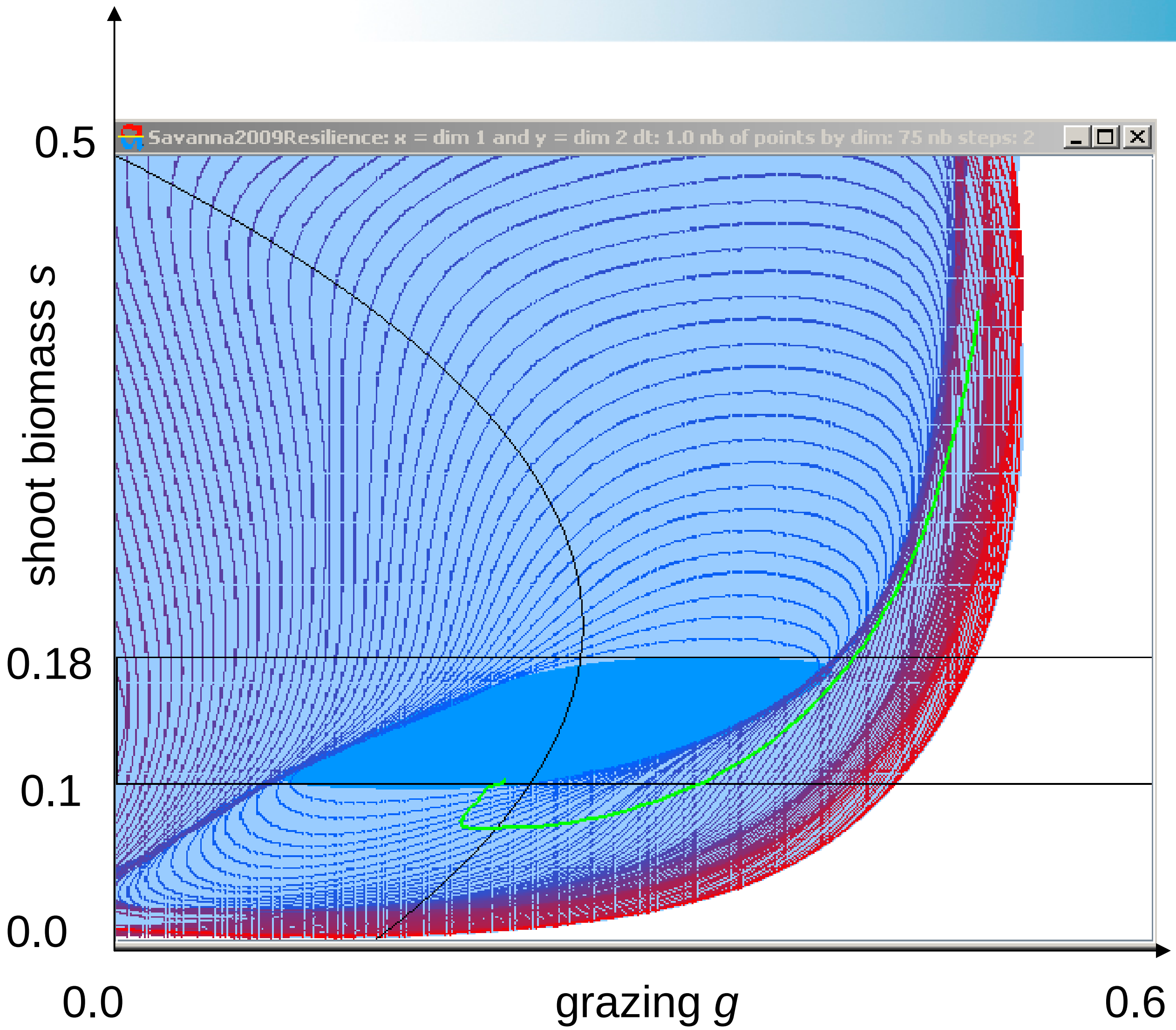
Viability kernel, with no stable equilibrium



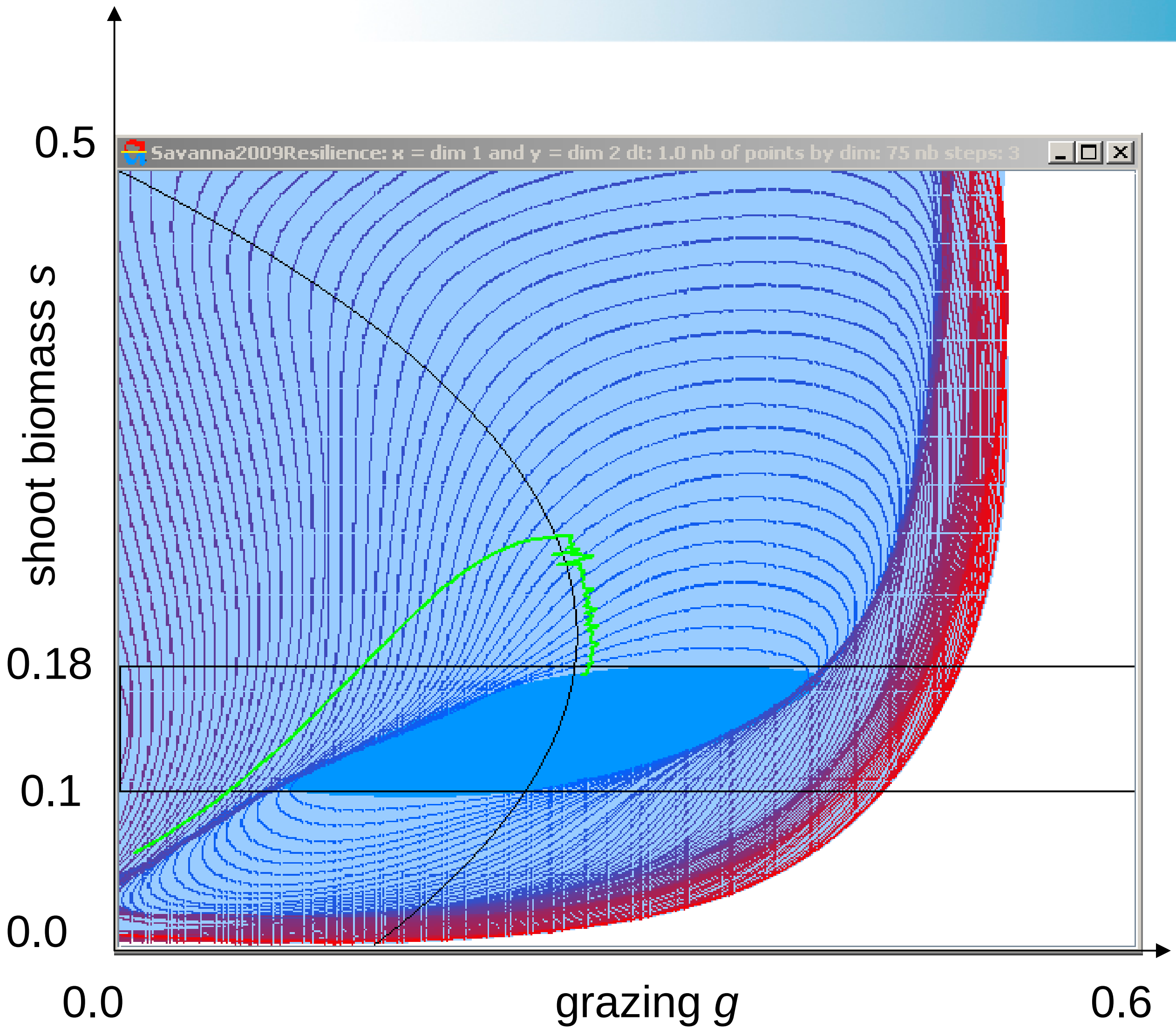
Resilience


0.5
Savanna2009Resilience: x = dim 1 and y = dim 2 dt: 1.0 nb of points by dim: 75 nb steps: 2
shoot biomass s
0.18
0.1
0.0
0.0
grazing g
0.6

Trajectory coming back to kernel



Trajectory coming back to kernel



Conclusion

- **The resilience based on viability can be seen as an extension of resilience based on attractors**
  - The « good » attractor is replaced by the viability kernel defined on the constraint set of the desired property
  - The attraction basin is replaced by the « capture basin » of the viability kernel (i.e. the points for which the exists a policy of action leading to the viability kernel)
- **Advantages**
  - Can include naturally an action in the approach and provides actions to make
  - Does not necessitate equilibirum in the dynamics.

Problem

- To compute viability kernels and resilience values, one must discretise the space
- When the dimension of the space grows, the number of points of the grid grows exponentially.
- The method cannot be applied on dynamical system with a state of many dimensions.